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VIRTUALLY FREE FACTORS OF PRO-p GROUPS

BY

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ABSTRACT

Let p be a prime number, G a pro-p group, and H a closed (topologically) finitely generated subgroup of G. We give conditions under which H is virtually a free factor of G, i.e., that there exists an open subgroup U of G such that U is the free pro-p product of H and some other subgroup of U. We prove that this happens if either G is a free pro-p group of any rank, or if G is a free pro-p product of finitely generated pro-p groups.

A la memoria de José Luis Rubio de Francia y de Pere Menal

1. Introduction

Marshall Hall proved in [4] that if F is a free abstract group and H a finitely generated subgroup of F, then any basis of the free group H can be extended to a basis of some subgroup N of finite index in F. In other words, $N = H * K$, for some subgroup K of N, where $H * K$ denotes the free product of H and K. Using the terminology of [1], we say that a group G is an M. Hall group if whenever H is a finitely generated subgroup of G, there exists some subgroup N of finite index in G such that N contains H and $N = H * K$, for some subgroup K of G. R. Burns extended the result of Hall to prove that the free product of two M. Hall groups is an M. Hall group (cf. [2]).

In this paper we consider the M. Hall property in the context of pro-p groups. First we study free pro-p groups in connection with that property, and we prove that every (topologically) finitely generated closed subgroup H of a free pro-p group F of arbitrary rank is a free factor of some open subgroup of F, i.e., there is an open subgroup U of F such that $U = H \amalg K$, where K is a closed subgroup

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of U, and ∐ denotes the free pro-p product of pro-p groups, i.e., the coproduct in the category of pro-p groups (see Corollary 3.5). This extends a result of A. Lubotzky (cf. [9], Th. 3.2), who proved this for free pro-p groups of finite rank.

Our main result (Theorem 4.5) deals with free pro-p products of pro-p groups, in connection with the M. Hall property. We prove an analogue for pro-p groups of the result of Burns mentioned above. Namely, consider (topologically) finitely generated pro-p groups G_i ($i = 1, \ldots, n$), with the property that for every (topologically) finitely generated subgroup H of G_i, there exists an open subgroup U of G_i such that $U = H \amalg K$, for some closed subgroup K; then we show that their free pro-p product $G_1 \amalg \cdots \amalg G_n$ satisfies the same property. This theorem generalizes a result of W. Herfort and the author (cf. [7]).

2. Notation and preliminaries

We follow the notation of [14] and [11]. Throughout the paper, p denotes a prime number. A pro-p group G is a projective limit of finite p-groups, over a directed set; or, equivalently, G is a compact, Hausdorff, totally disconnected topological group, whose open subgroups have an index which is a power of p. One says that a pro-p group G is generated by a subset X if G is the topological closure of the abstract subgroup of G generated by X. The pro-p group is *finitely generated*, if there exists a finite subset X of G that generates G in the above sense. The *Frattini subgroup* G^* of G is the intersection of all the maximal closed subgroups of G. It is easily seen that G/G^* is a vector space over the field $\mathbf{F}_p$ with p elements. The elements of G^* are characterized as the non-generators of G in the following sense: for a compact subset T of G, one has that T is a subset of G^* if and only if whenever $T \cup S$ generates G, so does S. Also $G^* = G^p[G,G]$, where G^p is the set of p-powers of the elements of G, and $[G,G]$ is the closure of the commutator subgroup of G. It is easily checked that a homomorphism $\varphi : G \to H$ of pro-p groups is surjective if and only if the induced map $\varphi : G/G^* \to H/H^*$ is surjective.

We say that a subset X of pro-p group G *converges to* 1, if every open subgroup U of G contains all but finitely many of the elements in X. Let F be a pro-p group and X a subset of F converging to 1; one says that F is a *free pro-p group* on the set X if the following universal property is satisfied: whenever G is a pro-p group, and $\theta : X \to G$ a map such that $\theta(X)$ converges to 1, there exists a unique continuous homomorphism $\bar{\theta} : F \to G$ which extends θ. See [11] or [14] for the basic properties of free pro-p groups. The free pro-p group on a set consisting of one element is $\mathbf{Z}_p$ (the additive group of p-adic integers).

Let $G_1, \ldots, G_n$ be pro-p groups; then, their *free pro-p product* G is defined to be their coproduct in the category of pro-p groups and continuous homomorphisms; we use the notation $G = G_1 \amalg \cdots \amalg G_n$ for such a product. We say that a closed subgroup H of a pro-p group G is a *free factor* of G, if there exists a closed subgroup K of G such that $G = H \amalg K$. See e.g., [7], [5], [10] for results on free products, and in particular for a proof of the following structure theorem that we will use throughout the paper.

THEOREM 2.1. *Let $G = G_1 \amalg \cdots \amalg G_n$ be the free pro-p product of the pro-p groups $G_1, \ldots, G_n$. Let H be a finitely generated closed subgroup of G. Then*

$$H = \left[\coprod_{i=1}^{n} \left(\coprod_{j} G_i^{\alpha(i,j)} \cap H \right) \right] \amalg F,$$

where: (a) *for each i, the set $\{\alpha(i,j) \mid j\}$ is a complete and irredundant set of double coset representatives of the subgroups G_i and H in G*; (b) *if $G_i\alpha(i,j)H = G_iH$, then $\alpha(i,j) = 1$*; *and* (c) *F is a free pro-p group.*

It easily follows from the above statement that F and each of the groups $G_i^{\alpha(i,j)} \cap H$ is finitely generated, and moreover, $G_i^{\alpha(i,j)} \cap H = 1$, for all but a finite number of the j's.

Finally, extending the terminology of [1], we say that a pro-p group group G is an *M. Hall pro-p group* if whenever A is a compact subset of G and H is a finitely generated closed subgroup of G that is disjoint from A, then H is a free factor of some open subgroup of G disjoint from A. Observe that Theorem 2.1 and an easy compactness argument imply that in the above definition one may assume that $A = \varnothing$.

Throughout the paper, unless otherwise explicitly stated, every homomorphism of pro-p groups is assumed to be continuous, and every subgroup of a pro-p group is supposed to be closed.

3. Free groups

We begin by stating a version of Lemma 3.1 in [7], which is mildly sharper than the original, and at the same time corrects an omission there. The proof is essentially the same as in [7], and we omit it.

LEMMA 3.1. *Let G be a pro-p group, T a compact subset of G, and H a finitely generated subgroup of G such that $H \cap T = \varnothing$. Then there exists an open normal subgroup N of G such that*

(i) $HN \cap T = \emptyset$,

(ii) $d(HN/N) = d(H)$, *where for a pro-p group* R, $d(R)$ *denotes the minimal number of generators of* R, *in a topological sense* (cf. §2),

(iii) *if* G *is finitely generated, every minimal set of generators of* H *can be extended to a minimal set of generators of* HN, *and*

(iv) $(HN)^* \cap H = H^*$. ∎

We state the following simple result for easy reference (cf. [5], Lemma 9.3 and [7], Lemma 3.8).

LEMMA 3.2. *Let* $A = B \amalg C$ *be a free pro-p product of pro-p groups* B *and* C. *Then*

(i) $B^* = B \cap A^*$, *and*

(ii) $A/A^* = BA^*/A^* \oplus CA^*/A^* \approx B/B^* \oplus C/C^*$. ∎

The following result extends Lemma 3.1 in [9].

PROPOSITION 3.3. *Let* F *be a free pro-p group, and let* H *be a subgroup of* F. *Then* H *is a free factor of* F *if and only if* $H \cap F^* = H^*$.

PROOF. By Lemma 3.2, if H is a free factor of F then $H \cap F^* = H^*$. Conversely, assume that $H \cap F^* = H^*$. Then H/H^* is a subspace of the $\mathbf{F}_p$-vector space F/F^*, and therefore there exists a (closed) direct complement C of H/H^* in F/F^* (cf. [5], Lemma 9.2). Let $\pi : F \to F/F^*$ be the canonical epimorphism. By Zorn's Lemma, there exists a minimal subgroup M of F such that $\pi(M) = C$. Then $\pi_{|M} : M \to C$ is a Frattini cover (cf. [3], p. 299), i.e., $\ker(\pi_{|M}) < M^*$, and so $M \cap F^* < M^*$. Therefore $M \cap F^* = M^*$, and hence π induces an isomorphism $M/M^* \approx C$. Set $G = H \amalg M$. Then $G/G^* \approx H/H^* \oplus M/M^*$, by Lemma 3.2(ii). The homomorphism $\alpha : G \to F$ induced by the inclusions $H, M \to F$ is surjective, since the induced map $\bar{\alpha} : G/G^* \to F/F^*$ is an isomorphism. Now, α has a right inverse $\beta : F \to G$, since F is a free group. However, β is also surjective, since the induced map $\bar{\beta}$ is the inverse of $\bar{\alpha}$, and hence surjective. Thus α is an isomorphism. ∎

COROLLARY 3.4. *Let* $F = F(X)$ *be a restricted free pro-p group on the set* X. *Then* $f_1, \ldots, f_n \in F$ *form part of a basis of* F (*converging to* 1) *iff they are* $\mathbf{F}_p$*-linearly independent modulo* F^*.

PROOF. Denote by H the subgroup of F generated by $f_1, \ldots, f_n$, and observe that if either $f_1, \ldots, f_n$ is part of a basis of F or if $f_1, \ldots, f_n$ are $\mathbf{F}_p$-linearly independent modulo F^*, then $f_1, \ldots, f_n$ is a basis for H. Consider the natural map

$H/H^* \to F/F^*$. Notice that $f_1, \ldots, f_n$ are linearly independent modulo F^* if and only if this map is an injection, and obviously this happens if and only if $H^* = H \cap F^*$. By Proposition 3.3 this condition is equivalent to saying that H is a free factor of F, or equivalently that a basis of H can be extended to a basis of F converging to 1. ■

The following corollaries generalize results of A. Lubotzky (cf. [9]), who proves them for free groups of finite rank. They can be proved using the arguments in [9], but we include here a different proof for Corollary 3.5.

COROLLARY 3.5. *Every free pro-p group F is an M. Hall pro-p group.*

PROOF. Let H be a finitely generated subgroup of F. By Lemma 3.1, there exists an open subgroup U of F such that $H \cap U^* = H^*$. Since U is a free pro-p group (cf. [11], Cor. 6.6, p. 236), the result follows from Proposition 3.3. ■

COROLLARY 3.6. *Let F be a free pro-p group, and let H be a finitely generated subgroup of F. If H contains a non-trivial normal subgroup in F, then H has finite index in F.* ■

4. Free products of pro-p groups

The next result, which is a generalization of Lemma 3.4 in [7], has been obtained jointly with M. Jarden.

LEMMA 4.1. *Let $G = G_1 \amalg \cdots \amalg G_n$ be a free pro-p product of pro-p groups, and let $\alpha(1), \ldots, \alpha(n) \in G$. Then $G = G_1^{\alpha(1)} \amalg \cdots \amalg G_n^{\alpha(n)}$.*

PROOF. For each open normal subgroup U of G, put $G_i(U) = G_i/G_i \cap U$. Then

$$G = \varprojlim_U \coprod_i G_i(U)$$

(cf. Lemma 4.2 in [7]). Denote by $\varphi_U : G \to \amalg_i G_i(U)$ the canonical projection. Put $\varphi_U(\alpha(i)) = \overline{\alpha(i)}$. By Lemma 3.4 in [7], $\amalg_i G_i(U) = \amalg_i G_i(U)^{\overline{\alpha(i)}}$. Hence

$$G = \varprojlim_U \coprod_i G_i(U) = \varprojlim_U \coprod_i G_i(U)^{\overline{\alpha(i)}} = \coprod_i \varprojlim_U G_i(U)^{\overline{\alpha(i)}} = \coprod_i G_i^{\alpha(i)}$$

(see [10], Prop. 1.6 for an explicit justification of the penultimate equality). ■

LEMMA 4.2. *Let $G_1, \ldots, G_n$ be pro-p groups and let $G = \coprod G_i$ be their free pro-p product. Let H be a subgroup of G, and suppose that H admits a decomposition of the form*

$$H = \left[\coprod_{i=1}^{n} \left(\coprod_{j} G_i^{\alpha(i,j)} \cap H \right) \right] \amalg F,$$

where $\alpha(i,j) \in G$, for each i, $G_i^{\alpha(i,j)} \cap H = 1$ for almost all j, and F is a free pro-p group. For each $i = 1, \ldots, n$ let S_i be any complete and irredundant set of representatives of the double cosets $G_i \backslash G / H$ of G with respect to G_i and H. Then for each i, $G_i^s \cap H = 1$ for almost all $s \in S_i$, and

$$H = \coprod_{i=1}^{n} \left[\coprod_{s \in S_i} G_i^s \cap H \right] \amalg F.$$

Proof. Observe first that if $j \neq k$, $G_i^{\alpha(i,j)} \cap H \neq 1$ and $G_i^{\alpha(i,k)} \cap H \neq 1$, then $G_i \alpha(i,j) H \neq G_i \alpha(i,k) H$. For otherwise there exist $g \in G_i$ and $h \in H$ such that $\alpha(i,j) = g\alpha(i,k)h$, and hence $G_i^{\alpha(i,j)} \cap H = (G_i^{\alpha(i,k)} \cap H)^h$, a contradiction (cf. Th. 2, [6]). Next if $G_i \alpha(i,j) H = G_i s H$, then $s = g\alpha(i,j)h$, $g \in G_i$ and $h \in H$, and so $G_i^s \cap H = (G_i^{\alpha(i,k)} \cap H)^h$; therefore, $G_i^s \cap H = 1$ iff $G_i^{\alpha(i,k)} \cap H = 1$. Let $\bar{S}_i = \{ s \in S_i \mid G_i^s \cap H \neq 1 \}$. Then

$$H = \left[\coprod_{i=1} \left(\coprod_{j}^{n} G_i^{\alpha(i,j)} \cap H \right) \right] \amalg F = \coprod_{i=1} \left[\coprod_{s \in \bar{S}_i} (G_i^s \cap H)^{h(s)} \right] \amalg F,$$

for some $h(s) \in H$. Hence, by Lemma 4.1,

$$H = \coprod_{i=1}^{n} \left[\coprod_{s \in S_i} G_i^s \cap H \right] \amalg F. \qquad \blacksquare$$

Lemma 4.3. *Subgroups of M. Hall pro-p groups are M. Hall pro-p groups.*

Proof. Let G be an M. Hall pro-p group, H a subgroup of G and K a finitely generated subgroup of H. By assumption, there exists an open subgroup U of G containing K with $U = K \amalg L$, for some subgroup L of U. Let $V = U \cap H$. Then V is open in H. Apply the Kurosh subgroup theorem (cf. [5] or [10]) to V as a subgroup of the free product $U = K \amalg L$, to get the desired result. $\blacksquare$

Lemma 4.4. *Consider pro-p groups $H \leq B \leq A$ such that H and B are finitely generated, and $H \cap B^* = H \cap A^* = H^*$. Assume that $B = R_0 \amalg R_1 \amalg F_1$ and $H = R_0 \amalg F$, where F and F_1 are free pro-p groups. If $R_1 A^* \leq R_0 A^*$, then H is a free factor of B.*

Proof. By assumption H/H^* is an $\mathbf{F}_p$-subspace of B/B^*. We claim that the subspaces FB^*/B^* and $(R_0 \amalg R_1)B^*/B^*$ of B/B^* have trivial intersection. For let $f \in F \setminus B^*$ and $r \in (R_0 \amalg R_1) \setminus B^*$ be such that $fr \in B^* \leq A^*$. Since $R_1 A^* \leq R_0 A^*$, $r = r_0 s$, where $s \in A^*$ and $r_0 \in R_0$. So $fr_0 \in H \cap A^* = H^*$. However, since

$H = R_0 \amalg F$, it follows from Lemma 3.2 that $f \in H^* \leq B^*$, a contradiction. This proves the claim. Let $x_1, \dots, x_s \in R_0 \amalg R_1$, and $f_1, \dots, f_t \in F$ be such that $x_1 B^*, \dots, x_s B^*$, and $f_1 B^*, \dots, f_t B^*$ form bases for the subspaces $(R_0 \amalg R_1)B^*/B^*$ and FB^*/B^* of B/B^*, respectively. Note that $f_1, \dots, f_t$ form a basis for F, since $FB^*/B^* \approx F/F \cap B^* = F/F \cap H \cap B^* = F/F \cap H^* \approx F/F^*$. Let $y_1, \dots, y_u \in B$ be such that $x_1 B^*, \dots, x_s B^*, f_1 B^*, \dots, f_t B^*, y_1 B^*, \dots, y_u B^*$ constitute a basis for B/B^*. Consider the subgroup S of B generated by $f_1, \dots, f_t, y_1, \dots, y_u$. By Lemma 3.2, rank $F_1 = t + u$. Define an epimorphism φ from B onto B that sends $R_0 \amalg R_1$ to $R_0 \amalg R_1$ identically, and sends F_1 onto S. Then φ is an isomorphism (cf. Prop. 7.6, p. 68 in [11]). It follows that $F_1 \approx S$ and $B = R_0 \amalg R_1 \amalg S = R_0 \amalg R_1 \amalg F \amalg \langle y_1, \dots, y_u \rangle = H \amalg R_1 \amalg \langle y_1, \dots, y_u \rangle$, as desired. ∎

THEOREM 4.5. *The free pro-p product of finitely many finitely generated M. Hall pro-p groups is an M. Hall pro-p group.*

Before we prove the theorem, we will state a consequence of it that extends Lemma 3.3 in [7].

COROLLARY 4.6. *Let $G = G_1 \amalg \cdots \amalg G_n$ be a free pro-p product where each G_i is either a finite p-group or isomorphic to $\mathbf{Z}_p$. Then every finitely generated subgroup H of G is a free factor of some open subgroup of G.*

PROOF OF THE THEOREM. Let $G = G_1 \amalg \cdots \amalg G_n$ be a free pro-p product of finitely generated M. Hall pro-p groups G_i, and let H be a finitely generated subgroup of G. We shall show that H is a free factor of some open subgroup of G. By Theorem 2.1

$$H = \left[\coprod_{i=1}^{n} \left(\coprod_{j} G_i^{\alpha(i,j)} \cap H \right) \right] \amalg F,$$

where the $\alpha(i,j)$'s are in G and form a complete and irredundant set of double coset representatives of G_i and H in G, and F is a free pro-p group; moreover $G_i^{\alpha(i,j)} \cap H = 1$ for almost all j's (say $G_i^{\alpha(i,j)} \cap H \neq 1$ if and only if $j = 1, \dots, r(i)$), and F is a free pro-p group of finite rank. Let N be an open normal subgroup of G such that for each $i = 1, \dots, n$, and $1 \leq j,k \leq r(i)$, $G_i\alpha(i,j)HN \neq G_i\alpha(i,k)HN$. Since HN is an open subgroup of G, there are only finitely many double cosets of G_i and HN in G. It follows then from Theorem 2.1 applied to HN, and Lemma 4.2, that

$$(*)\quad HN = \left[\coprod_{i=1}^{n} \left(\coprod_{j=1}^{r(i)} G_i^{\alpha(i,j)} \cap HN \right) \right] \amalg \left[\coprod_{i=1}^{n} \left(\coprod_{k=1}^{t(i)} G_i^{\beta(i,k)} \cap HN \right) \right] \amalg F(N),$$

where $\beta(i,j) \in G$, $\{\alpha(i,j), \beta(i,k) \mid j = 1,\dots,r(i),\ k = 1,\dots,t(i)\}$ are representatives of disjoint double cosets of G_i and HN in G, and $F(N)$ is a free pro-p group of finite rank. Using Lemma 3.1, we choose N to be such that, in addition, $(HN)^* \cap H = H^*$.

By Lemma 4.3, the finitely generated group $G_i^{\alpha(i,j)} \cap H$ is a free factor of an open subgroup U_{ij} of $G_i^{\alpha(i,j)} \cap HN$; say $U_{ij} = (G_i^{\alpha(i,j)} \cap H) \amalg T_{ij}$. We claim that if one chooses N small enough, then $G_i^{\alpha(i,j)} \cap H$ is in fact a free factor of $G_i^{\alpha(i,j)} \cap HN$ itself. For choose an open normal subgroup S of G with $S \leq N$ such that

$$(G_i^{\alpha(i,j)} \cap HN) \cap HS = G_i^{\alpha(i,j)} \cap HS \leq U_{ij} \qquad \text{for each } i \text{ and } j;$$

then apply Theorem 2.1 again to HS as a subgroup of the above free product decomposition (*), and then to $G_i^{\alpha(i,j)} \cap HS$ as a subgroup of the free product $U_{ij} = (G_i^{\alpha(i,j)} \cap H) \amalg T_{ij}$, to see that $G_i^{\alpha(i,j)} \cap H$ is a free factor of $G_i^{\alpha(i,j)} \cap HS$; finally substitute N by S, proving the claim. Then we can rewrite HN as

$$HN = \left[\coprod_{i=1}^{n} \left(\coprod_{j=1}^{r(i)} (G_i^{\alpha(i,j)} \cap H) \amalg T_{ij}\right)\left(\coprod_{k=1}^{t(i)} G_i^{\beta(i,k)} \cap HN\right)\right] \amalg F(N).$$

Observe that since HN is finitely generated, so is each T_{ij} and $G_i^{\beta(i,k)} \cap HN$. Therefore, the Frattini subgroups $(T_{ij})^*$ and $(G_i^{\beta(i,k)} \cap HN)^*$ are open subgroups of T_{ij} and $G_i^{\beta(i,k)} \cap HN$ respectively. On the other hand, $H \cap T_{ij} = 1 = H \cap G_i^{\beta(i,k)} \cap HN$, for all $i = 1,\dots,n$, $j = 1,\dots,r(i)$, and $k = 1,\dots,t(i)$. Therefore, there exists an open normal subgroup M of G with $M \leq N$ and such that $HM \cap T_{ij} \leq T_{ij}^*$ and $HM \cap G_i^{\beta(i,k)} \cap HN = G_i^{\beta(i,k)} \cap HM \leq (G_i^{\beta(i,k)} \cap HN)^*$. Apply again Theorem 2.1 to HM as a subgroup of the above free product decomposition of HN, to get

$$\begin{aligned} HM = \Bigg\{\coprod_{i=1}^{n} \Bigg[\coprod_{j=1}^{r(i)} &\left(\coprod_{u} (G_i^{\alpha(i,j)} \cap H)^{\delta(i,j,u)} \cap HM\right) \amalg \left(\coprod_{v} T_{ij}^{\gamma(i,j,v)} \cap HM\right) \\ &\amalg \left(\coprod_{k=1}^{t(i)} \coprod_{z} (G_i^{\beta(i,k)} \cap HN)^{\epsilon(i,k,z)} \cap HM\right)\Bigg]\Bigg\} \amalg F_1, \end{aligned}$$

where F_1 is a free pro-p group, $\delta(i,j,u), \gamma(i,k,v), \epsilon(i,k,z)$ are representatives of the double cosets of $G_i^{\alpha(i,j)} \cap HN$ and HM in HN, of T_{ij} and HM in HN, and of $G_i^{\beta(i,k)} \cap HN$ and HM in HN, respectively; moreover, as usual, we take 1 to be the representative of the double cosets that contain 1, so that for each i,j, $G_1^{\alpha(i,j)} \cap H$ is a factor appearing in the above decomposition of HM. Since N was chosen so that $(HN)^* \cap H = H^*$, we also have $(HM)^* \cap H = H^*$, and so H/H^* is an $\mathbf{F}_p$-subspace of $HM/(HM)^*$. Set

$$R_0 = \left[\coprod_{i=1}^{n} \left(\coprod_{j=1}^{r(i)} G_i^{\alpha(i,j)} \cap H \right) \right].$$

Then $H = R_0 \amalg F$, and $HM = R_0 \amalg R_1 \amalg F_1$, where

$$R_0 \amalg R_1 = \coprod_{i=1}^{n} \left[\coprod_{j} \left(\coprod_{u} (G_i^{\alpha(i,j)} \cap H)^{\delta(i,j,u)} \cap HM \right) \amalg \left(\coprod_{v} T_{ij}^{\gamma(i,j,v)} \cap HM \right) \right.$$
$$\left. \amalg \left(\coprod_{k} \coprod_{z} (G_i^{\beta(i,k)} \cap HN)^{\epsilon(i,k,z)} \cap HM \right) \right].$$

Observe that $(G_i^{\alpha(i,j)} \cap H)^{\delta(i,j,u)} \cap HM \equiv G_i^{\alpha(i,j)} \cap H$, modulo $(HN)^*$; $T_{ij}^{\gamma(i,j,v)} \cap HM \equiv T_{ij} \cap HM \equiv 1$, modulo $(HN)^*$; and that $((G_i^{\beta(i,k)} \cap HN)^{\epsilon(i,k,z)} \cap HM) \equiv G_i^{\beta(i,k)} \cap HM \equiv 1$, modulo $(HN)^*$. Hence the conditions of Lemma 4.4 are satisfied, where HN and HM play the rôles of A and B respectively. Therefore, H is a free factor of HM as desired. ■

5. Final remarks

Before we state the next result, we recall the concept of free product of two groups amalgamating a common subgroup in the context of pro-p groups (see [12] for details). Let A and B be pro-p groups with a common subgroup C. Consider the push-out diagram

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & G \end{array}$$

in the category of pro-p groups. One says that G is the free product of A and B amalgamating C, and we write $G = A \amalg_C B$, if the canonical maps $A \to G$ and $B \to G$ are monomorphisms. It turns out that if A and B are finite p-groups (or, more generally, countably generated pro-p groups) then $G = A \amalg_C B$ iff $A *_C B$ (the free product with amalgamation, as abstract groups) is a residually finite p-group; and in fact, then G is the pro-p completion of $A *_C B$ (cf. Th. 3.1 in [12]).

Proposition 5.1. *Let A and B be finite p-groups with a common subgroup $C \neq 1$, and $A \neq C \neq B$. Assume that the free product of A and B amalgamating C, $G = A \amalg_C B$ exists. Then G contains finitely generated subgroups that are not free factors of any open subgroup of G.*

Proof. Suppose not. Choose subgroups A' and B' of A and B respectively such that $C < A'$, $C < B'$, $(A' : C) = p$ and $(B' : C) = p$. Then C is a normal subgroup of both A' and B'. Let $G' = \langle A', B' \rangle$ be the subgroup of G generated by A' and B'. Then $C \rhd G'$. By Lemma 4.3, C is a free factor of an open subgroup U of G'. Say $U = C \amalg C'$. Now by Lemma 2 in [13], the abstract subgroup H generated by A' and B' is the free product with amalgamation $H = A' *_C B'$. Note $(U:C) = \infty$, since U is open in G' and $(G':C) \geq (H:C) \geq \infty$. Therefore, $C' \neq 1$, and since $C \rhd U$, we have that C' normalizes C. However, by Th. 2 in [6], the only elements of U normalizing C are those of C. This contradiction implies the result. ∎

Conjecture 5.2. The only finitely generated M. Hall pro-p groups indecomposable with respect to free pro-p products are either finite p-groups or $\mathbf{Z}_p$.

Conjecture 5.3. Theorem 4.5 is valid even if the free factors are not necessarily finitely generated.

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